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Quantum electrodynamics in an analytic representation

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Abstract. The analytic representation of quantum field theory suggested in a previous paper by Valatin and the author is further investigated, by working out its details for quantum electrodynamics. The coupled equations for Green's functions are discussed and the lowest order contributions to Compton and Møller scatterings are calculated. Through the Fourier transforms, the expressions show close agreement with the results of the conventional approach. A nonlocal modification of the interaction kernel is studied. This leads to finite self-energy, vacuum polarization and vertex operator expressions, and in the limit, to a simple regularization of quantum electrodynamics.

1. Introduction

This paper is a continuation of a previous paper by the author and Valatin (1969 to be referred to as I), in which an analytic representation of quantum field theory has been proposed. The formulation is based on the connection between the Hilbert space of quadratically integrable functions of real variables q and a Hilbert space of analytic functions of complex variables z, originally established by Bargmann (1961, 1967). The operators in this formulation are proper operators for which the δ-functions of the q-space commutation relations are replaced by analytic kernel functions in z-space and a reference to distributions is therefore avoided. Local interaction terms in q-space take a 'nonlocal' appearance in z-space with the presence of an interaction kernel W in each interaction term which is a function of three sets of complex variables and depends on an arbitrarily chosen length a. For a four-dimensional formulation one has to work in euclidean metric and the field equations can be derived directly from an action integral in euclidean z-space with the application of a variational calculus in which the variations are restricted to analytic functions only.

The application of this analytic representation of quantum field theory to electrodynamics forms the subject matter of the present paper. To study the interaction of a Dirac field with an electromagnetic field, field equations as well as coupled equations for Green's functions in euclidean z-space are derived and scattering amplitudes in momentum space are calculated using a simple relationship between the Green's functions and the scattering amplitudes. The results obtained are the same as the ones from the conventional theory and are independent of the arbitrarily chosen length a which is an essential feature of the z-space formulation.

The interesting aspect of the z-space formulation of quantum field theory is the presence of the interaction kernel W at each vertex. Since the formulation itself is independent of the structure of W, one might consider the analytic representation of quantum field theory as a more general theory with the theory corresponding to the local q-space interaction embedded in it. It will be of interest then to study different structures of W and the physical conclusions which result from them. We propose to show that a simple modification of this interaction kernel leads to finite electron self-energy, vacuum polarization and vertex operator expressions and, in the limit, to a simple regularization of electrodynamics.

2. Field equations

The analytic representation of quantum field theory, as has been suggested in I, is based on the connection between the Hilbert space $\mathscr{H}$ of square integrable functions $\varphi(q)$ of real variables q and a Hilbert space $\mathscr{F}$ of analytic functions $f(z)$ of complex variables z

$$f(z) = \int \mathrm{d}q\, A(z, q)\,\varphi(q) \tag{1a}$$

where the complex variable z is chosen as

$$z = x - \mathrm{i}a^2k \qquad \bar{z} = x + \mathrm{i}a^2k \tag{1b}$$

x and k being the average position and momentum of a wave packet of width a and the transformation kernel A is given by

$$A(z, q) = (\pi a^2)^{-1/4} \exp\left(-\frac{1}{2a^2}(z-q)^2\right). \tag{1c}$$

The scalar product of two functions f and $f' \in \mathscr{F}$ is defined by the integral

$$(f, f') = \int \mathrm{d}\mu(z)\overline{f(z)}f'(z) \tag{1d}$$

with

$$\mathrm{d}\mu(z) = \frac{\mathrm{d}x\,\mathrm{d}(-k)}{2\pi}\exp\left(\frac{1}{4a^2}(z-\bar{z})^2\right) \tag{1e}$$

and is equal to the scalar product

$$(\varphi, \varphi') = \int \mathrm{d}q\,\varphi^*(q)\,\varphi'(q) \tag{2a}$$

of the corresponding functions φ and $\varphi' \in \mathscr{H}$. The inverse mapping of $\mathscr{F}$ onto $\mathscr{H}$ is defined by

$$\varphi(q) = \int \mathrm{d}\mu(z)\, A(\bar{z}, q) f(z) \tag{2b}$$

and the transformation kernel A, with properties

$$\int \mathrm{d}\mu(z)\, A(z, q)\, A(\bar{z}, q') = \delta(q-q') \tag{3a}$$

$$\int \mathrm{d}q A(z, q) A(\bar{z}, q) = \exp\left(-\frac{1}{4a^2}(z-\bar{z}')^2\right) = U(z-\bar{z}') \tag{3b}$$

establishes a one-to-one correspondence between $\mathscr{H}$ and $\mathscr{F}$. This Hilbert space $\mathscr{F}$ of analytic functions $f(z)$ of complex variables z is a modification of the Hilbert space of analytic functions originally introduced by Bargmann (1961, 1967).

It then follows from the above discussion that any integral operator kernel $K(q, q')$ of an operator K operating on functions $\varphi(q)$ corresponds in z representation to a kernel $K(z, \bar{z}')$

$$K(z, \bar{z}') = \int \mathrm{d}q \int \mathrm{d}q' A(z, q)\, K(q, q')\, A(\bar{z}', q') \tag{4a}$$

so that the δ-function kernel $\delta(q-q')$ of the unit operator is transformed into the unit kernel $U(z-\bar{z}')$ given by (3*b*). This unit kernel is an analytic function of the complex variables z and $\bar{z}'$ and plays the role of the reproducing kernel

$$f(z) = \int d\mu(z')\, U(z-\bar{z}') f(z') \tag{4b}$$

in the Hilbert space of analytic functions.

Thus one sees that in a theory formulated in this Hilbert space of analytic functions the δ-functions will be replaced by analytic unit functions U and a reference to distributions will be avoided. This is the underlying idea in the analytic representation of quantum field theory suggested in I. In order to use the Hilbert space of analytic functions for a four-dimensional theory one is forced to work in euclidean metric. Schwinger (1958, 1959), Nakano (1959) and Symanzik (1966) have studied field theory in euclidean space and have shown that one can analytically continue from Minkowski space field theory to euclidean space field theory with

$$q_0 = t \rightarrow q_4 = iq_0 = it$$

q_4 being considered real. Then corresponding to a field

$$\Psi(q) = \Psi(\mathbf{q}, q_4) = \Psi(q_1, q_2, q_3, q_4)$$

in euclidean q-space, one can define a field $\Psi(z)$ in euclidean z-space by

$$\Psi(z) = \Psi(\mathbf{z}, z_4) = \Psi(z_1, z_2, z_3, z_4) = \int d^4q\, A^{(4)}(z, q)\Psi(q) \tag{5a}$$

with

$$d^4q = dq_1\, dq_2\, dq_3\, dq_4 \tag{5b}$$

and

$$A^{(4)}(z, q) = A(z_1, q_1) A(z_2, q_2) A(z_3, q_3) A(z_4, q_4). \tag{5c}$$

The fields $\Psi(z)$ in the four-dimensional z-space will depend on the four complex variables $z = z_1, z_2, z_3, z_4$, and the relevant functions will contain the euclidean invariant form $z^2 = z_1^2+z_2^2+z_3^2+z_4^2$.

We now proceed to obtain the field equations for the local interaction of an electron field with an electromagnetic field. By a straight transformation of the action integral in euclidean q-space, in which the transformations of the fields from q-space to z-space are done according to (5*a*), one gets an action integral in the four dimensional euclidean z-space

$$\begin{aligned} I = &\int d^4\mu(z)\Big\{-\frac{1}{2i}\Big(\bar{\Psi}(\bar{z})\,\gamma_\lambda \frac{\partial}{\partial z_\lambda}\Psi(z) - \frac{\partial}{\partial \bar{z}_\lambda}\bar{\Psi}(\bar{z})\gamma_\lambda\Psi(z)\Big) - m\,\bar{\Psi}(\bar{z})\Psi(z)\Big\} \\ &+\int d^4\mu(z)\{-\tfrac{1}{2}A_\lambda(\bar{z})\,\Box_z A_\lambda(z)\} \\ &+e\int \dots \int d^4\mu(z)\, d^4\mu(z')\, d^4\mu(z'')W^{(4)}(z, \bar{z}', \bar{z}'')\bar{\Psi}(\bar{z})\gamma_\lambda\Psi(z')A_\lambda(z'') \end{aligned} \tag{6a}$$

with

$$d^4\mu(z) = d\mu(z_1)\, d\mu(z_2)\, d\mu(z_3)\, d\mu(z_4) \tag{6b}$$

$$-\Box_z = \frac{\partial^2}{\partial z_1^2}+\frac{\partial^2}{\partial z_2^2}+\frac{\partial^2}{\partial z_3^2}+\frac{\partial^2}{\partial z_4^2} \tag{6c}$$

and

$$[\gamma_\lambda, \gamma_\nu]_+ = 2g_{\lambda\nu} \qquad \lambda, \nu = 1, 2, 3, 4 \qquad \gamma_4 = i\gamma_0 \tag{6d}$$

$$g_{\lambda\nu} = -1 \qquad \text{for } \lambda = \nu$$
$$= 0 \qquad \text{for } \lambda \neq \nu.$$

We should mention here that the real field $A(q)$ can be transformed either into $A(z)$ or into $A(\bar{z})$ and therefore the integrands of (6*a*) can be symmetrized with respect to z and $\bar{z}$. The action integral has the same form as the one in euclidean q-space except for the interaction term which appears in a nonlocal form described by the interaction kernel W

$$W^{(4)}(z, z', z'') = \int d^4q\, A^{(4)}(z, q)\, A^{(4)}(z', q)\, A^{(4)}(z'', q)$$

$$= \{\pi(\tfrac{3}{2}a)^2\}^{-1} \exp\left(-\frac{(z-z')^2+(z'-z'')^2+(z''-z)^2}{6a^2}\right). \tag{6e}$$

We now apply a variational calculus in which the variations are restricted to analytic functions of complex variables z, that is, variations $\delta\varphi(z)$ of functions $\varphi(z)$ satisfy the identity

$$\delta(\varphi)z = \int d^4\mu(z')\, U^{(4)}(z-\bar{z}')\, \delta\varphi(z') \tag{7a}$$

and the unit kernel

$$U^{(4)}(z-\bar{z}') = U(z_1-\bar{z}_1')U(z_2-\bar{z}_2')U(z_3-\bar{z}_3')U(z_4-\bar{z}_4') \tag{7b}$$

appears instead of the δ-functions in expressions like

$$\frac{\delta\varphi(z)}{\delta\varphi(z')} = U^{(4)}(z-\bar{z}'). \tag{7c}$$

Then from the equations

$$\frac{\delta I}{\delta\bar{\Psi}(\bar{z})} = 0 \qquad \frac{\delta I}{\delta\Psi(z)} = 0 \qquad \frac{\delta I}{\delta A_\lambda(z)} = 0 \tag{7d}$$

one gets, within the framework of a c-number theory,

$$(i\nabla_z + m)\Psi(z) = e \iint d^4\mu(z')\, d^4\mu(z'') W^{(4)}(z, \bar{z}', \bar{z}'') A_\lambda(z')\gamma_\lambda\Psi(z'') \tag{8a}$$

$$\bar{\Psi}(\bar{z})(-i\nabla_{\bar{z}} + m) = e \iint d^4\mu(z')\, d^4\mu(z'') W^{(4)}(\bar{z}, z', \bar{z}'')\bar{\Psi}(\bar{z}')\gamma_\lambda A_\lambda(z'') \tag{8b}$$

$$\Box_z A_\lambda(z) = -e \iint d^4\mu(z')\, d^4\mu(z'') W^{(4)}(z, z', \bar{z}'')\bar{\Psi}(\bar{z}')\gamma_\lambda\Psi(z'') \tag{8c}$$

with

$$-\nabla_z = \gamma_1\frac{\partial}{\partial z_1} + \gamma_2\frac{\partial}{\partial z_2} + \gamma_3\frac{\partial}{\partial z_3} + \gamma_4\frac{\partial}{\partial z_4}. \tag{8d}$$

Equations (8*a*, *b*, *c*) are the z-space field equations for an electron interacting with an electromagnetic field. The presence of the interaction kernel W gives the equations a 'nonlocal' appearance and as long as one chooses W given by (6*e*), the equations correspond to a local interaction in q-space. A change in the structure of W

may lead to a nonlocal interaction in q-space and we propose to show in the last section of the paper how a simple modification of W results in a simple regularization of electrodynamics.

3. Green's function equations

Starting from an action integral one can obtain Green's function equations of quantum electrodynamics by means of a simple formal quantization with the help of external source functions (Symanzik 1954, 1960, Valatin 1955). To obtain Green's function equations in z-space we introduce external source functions $\xi(\bar{z})$, $\eta(z)$ and $J(z)$, which are analytic functions of z and $\bar{z}$, by defining an action integral

$$I' = I + \int d^4\mu(z)\{\xi(\bar{z})\Psi(z) + \bar{\Psi}(\bar{z})\eta(z) + J_\lambda(\bar{z})A_\lambda(z)\} \tag{9a}$$

with I given by (6*a*). A and J being neutral fields, one can symmetrize their products in the integrand of (9*a*) with respect to z and $\bar{z}$. The c-number field equations obtained from the above action integral

$$\frac{\delta I}{\delta\bar{\Psi}(\bar{z})} = -\eta(z) \qquad \frac{\delta I}{\delta\Psi(z)} = -\xi(\bar{z}) \qquad \frac{\delta I}{\delta A_\lambda(\bar{z})} = -J_\lambda(z) \tag{9b}$$

have in addition to the terms of (7*d*) additional inhomogeneous source terms. The generating functional χ

$$\chi = \chi[\bar{\xi}, \eta, J] \tag{9c}$$

of the many-particle Green's functions satisfies the equations obtained by applying the inhomogeneous equations (9*b*) on χ, after replacing the fields by functional differential operators according to

$$\Psi(z) \to i\frac{\delta}{\delta\xi(\bar{z})} \qquad \bar{\Psi}(\bar{z}) \to \frac{1}{i}\frac{\delta}{\delta\eta(z)} \qquad A_\lambda(z) \to i\frac{\delta}{\delta J_\lambda(\bar{z})}. \tag{9d}$$

With this prescription we then get the following equations for the generating functional

$$\left((i\nabla_z + m)i\frac{\delta}{\delta\xi(\bar{z})} + e\int\int d^4\mu(z')\, d^4\mu(z'')W^{(4)}(z, \bar{z}', \bar{z}'')\gamma_\lambda \frac{\delta}{\delta\xi(\bar{z}')}\frac{\delta}{\delta J_\lambda(\bar{z}'')} + \eta(z)\right)\Big|\, \chi[\xi, \eta, J] = 0 \tag{10a}$$

$$\left(\frac{1}{i}\frac{\delta}{\delta\eta(z)}(-i\nabla_{\bar{z}} + m) - e\int\int d^4\mu(z')\, d^4\mu(z'')W^{(4)}(\bar{z}, z', \bar{z}'')\frac{\delta}{\delta\eta(z')}\gamma_\lambda\frac{\delta}{\delta J_\lambda(\bar{z}'')} + \xi(\bar{z})\right)\Big|\, \chi[\xi, \eta, J] = 0 \tag{10b}$$

$$\left(\Box_z i\frac{\delta}{\delta J_\lambda(z)} + e\int\int d^4\mu(z')\, d^4\mu(z'')W^{(4)}(z, z', \bar{z}'')\frac{\delta}{\delta\eta(z')}\gamma_\lambda\frac{\delta}{\delta\xi(\bar{z}'')} + J_\lambda(z)\right)\Big|\, \chi[\xi, \eta, J] = 0. \tag{10c}$$

With the power series expansion of the functional χ

$$\chi[\xi, \eta, J] = \sum_{m,n,p,q} \frac{(-\mathrm{i})^{m+n+p+q}}{m!n!(p+q)!} \int \ldots \int \mathrm{d}^4\mu(z_1) \ldots \mathrm{d}^4\mu(z_m)\, \mathrm{d}^4\mu(z_1') \ldots \mathrm{d}^4\mu(z_n')\mathrm{d}^4\mu(Z_1)$$
$$\ldots \mathrm{d}^4\mu(Z_p)\, \mathrm{d}^4\mu(Z_1') \ldots \mathrm{d}^4\mu(Z_q')\xi(\bar{z}_1) \ldots \xi(\bar{z}_m)J_{\lambda_1}(Z_1) \ldots J_{\lambda_p}(Z_p)$$
$$\times G_{\lambda_1\ldots\lambda_p\nu_1\ldots\nu_q}(z_1, \ldots z_m, \bar{z}_1', \ldots \bar{z}_n'; Z_1, \ldots Z_p, Z_1', \ldots Z_q')$$
$$\times J_{\nu_q}(Z_q') \ldots J_{\nu_1}(Z_1')\eta(z_n') \ldots \eta(z_1') \tag{10d}$$

in which z and Z have been used for electron and photon coordinates respectively to avoid confusion and which has been symmetrized with respect to Z and $\bar{Z}$ using the fact that the sources J are neutral, one gets from (10a, b, c), after equating the coefficients to zero,

$$(\mathrm{i}\nabla_z + m)\mathrm{G}_{\lambda_1\ldots\lambda_p\nu_1\ldots\nu_q}(z, z_1, \ldots z_m, \bar{z}_1', \ldots \bar{z}_n'; Z_1, \ldots Z_p, Z_1', \ldots Z_q')$$
$$= \sum_{j=1}^{n} U^{(4)}(z-\bar{z}_j')\mathrm{G}_{\lambda_1\ldots\lambda_p\nu_1\ldots\nu_q}(z_1, \ldots z_m, \bar{z}_1', \ldots \hat{\bar{z}}_j', \ldots \bar{z}_n'; Z_1, \ldots Z_p, Z_1',$$
$$\ldots Z'_q) + e\iint \mathrm{d}^4\mu(z')\, \mathrm{d}^4\mu(Z)W^{(4)}(z, \bar{z}', Z)\gamma_\lambda \mathrm{G}_{\lambda\lambda_1\ldots\lambda_p\nu_1\ldots\nu_q}(z', z_1, \ldots z_m, \bar{z}_1',$$
$$\ldots \bar{z}_n'; Z, Z_1, \ldots Z_p, Z_1', \ldots Z_q') \tag{11a}$$
$$\mathrm{G}_{\lambda_1\ldots\lambda_p\nu_1\ldots\nu_q}(z_1, \ldots z_m, \bar{z}, \bar{z}_1', \ldots \bar{z}_n'; Z_1, \ldots Z_p, Z_1', \ldots Z_q')(-\mathrm{i}\nabla_{\bar{z}} + m)$$
$$= \sum_{j=1}^{m} \mathrm{G}_{\lambda_1\ldots\lambda_p\nu_1\ldots\nu_q}(z_1, \ldots \hat{z}_j, \ldots z_m, \bar{z}_1', \ldots \bar{z}_n'; Z_1, \ldots Z_p, Z_1', \ldots Z_q')$$
$$\times U^{(4)}(z_j - \bar{z}) + e\iint \mathrm{d}^4\mu(z')\, \mathrm{d}^4\mu(Z')\mathrm{G}_{\lambda_1\ldots\lambda_p\nu\nu_1\ldots\nu_q}(z_1, \ldots z_m, \bar{z}', \bar{z}_1', \ldots \bar{z}_n'; Z_1,$$
$$\ldots Z_p, Z', Z_1', \ldots Z_q')\gamma_\nu W^{(4)}(\bar{z}, z', Z') \tag{11b}$$
$$\Box_Z \mathrm{G}_{\lambda\lambda_1\ldots\lambda_p\nu_1\ldots\nu_q}(z_1, \ldots z_m, \bar{z}_1', \ldots \bar{z}_n'; Z, Z_1, \ldots Z_p, Z_1', \ldots Z_q')$$
$$= -\sum_{j=1}^{q} g_{\lambda\nu_j} U^{(4)}(Z - Z_j')\mathrm{G}_{\lambda_1\ldots\lambda_p\nu_1\ldots\hat{\nu}_j\ldots\nu_q}(z_1, \ldots z_m, \bar{z}_1', \ldots \bar{z}_n'; Z_1,$$
$$\ldots Z_p, Z_1', \ldots \hat{Z}_j', \ldots Z_q') - e\iint \mathrm{d}^4\mu(z')\, \mathrm{d}^4\mu(z'')W^{(4)}(Z, \bar{z}', z'')\gamma_\lambda$$
$$\times \mathrm{G}_{\lambda_1\ldots\lambda_p\nu_1\ldots\nu_q}(z', z_1, \ldots z_m, \bar{z}'', \bar{z}_1', \ldots \bar{z}_n', Z_1, \ldots Z_p, Z_1', \ldots Z_q') \tag{11c}$$

where a hat on the top of any term, like $\hat{\bar{z}}_j$, means that the particular term is missing from the expression. The function G is the many-particle Green's function of electrodynamics in z-space and corresponds to the time-ordered Green's function in euclidean q-space.

If we now use the free electron and photon Green's functions S and D

$$(\mathrm{i}\nabla_z + m)\,\mathrm{S}(z, \bar{z}') = U^{(4)}(z - \bar{z}') \tag{11d}$$
$$\mathrm{S}(z, \bar{z}')\,(-\mathrm{i}\nabla_{\bar{z}'} + m) = U^{(4)}(z - \bar{z}') \tag{11e}$$
$$\mathrm{D}_{\lambda\nu}(Z, Z') = -g_{\lambda\nu}\mathrm{D}(Z, Z')$$
$$\Box_Z \mathrm{D}(Z, Z') = U^{(4)}(Z - Z') \tag{11f}$$

which have been discussed in detail in I, we get from (11a, b, c) three sets of infinitely coupled equations for the many-particle Green's functions G

$$G_{\lambda_1\ldots\lambda_p\nu_1\ldots\nu_q}(z, z_1, \ldots z_m, \bar{z}_1', \ldots \bar{z}_n'; Z_1, \ldots Z_p, \bar{Z}_1', \ldots \bar{Z}_q')$$
$$= \int d^4\mu(z') \sum_{j=1}^{n} S(z, \bar{z}')U^{(4)}(z' - \bar{z}_j)G_{\lambda_1\ldots\lambda_p\nu_1\ldots\nu_q}(z_1, \ldots z_m, \bar{z}_1', \ldots \hat{\bar{z}}_j',$$
$$\ldots \bar{z}_n'; Z_1, \ldots Z_p, \bar{Z}_1', \ldots \bar{Z}_q') + e\int \ldots \int d^4\mu(z')\, d^4\mu(z'')\, d^4\mu(Z)\, S(z, \bar{z}')$$
$$\times W^{(4)}(z', \bar{z}'', \bar{Z})\gamma_\lambda G_{\lambda\lambda_1\ldots\lambda_p\nu_1\ldots\nu_q}(z'', z_1, \ldots z_m, \bar{z}_1', \ldots \bar{z}_n'; Z, Z_1, \ldots Z_p, \bar{Z}_1',$$
$$\ldots \bar{Z}_q') \tag{12a}$$

$$G_{\lambda_1\ldots\lambda_p\nu_1\ldots\nu_q}(z_1, \ldots z_m, \bar{z}, \bar{z}_1', \ldots \bar{z}_n'; Z_1, \ldots Z_p, \bar{Z}_1', \ldots \bar{Z}_q')$$
$$= \int d^4\mu(z') \sum_{j=1}^{m} G_{\lambda_1\ldots\lambda_p\nu_1\ldots\nu_q}(z_1, \ldots \hat{z}_j, \ldots z_m, \bar{z}_1', \ldots \bar{z}_n'; Z_1, \ldots Z_p, \bar{Z}_1', \ldots \bar{Z}_q')$$
$$\times U^{(4)}(z_j - \bar{z}')S(z', \bar{z}) + e\int \ldots \int d^4\mu(z')\, d^4\mu(z'')\, d^4\mu(Z')G_{\lambda_1\ldots\lambda_p\nu\nu_1\ldots\nu_q}(z_1,$$
$$\ldots z_m, \bar{z}', \bar{z}_1', \ldots \bar{z}_n'; Z_1, \ldots Z_p, \bar{Z}', \bar{Z}_1', \ldots \bar{Z}_q')$$
$$\times \gamma_\nu W^{(4)}(z', \bar{z}'', Z')\, S(z'', \bar{z}) \tag{12b}$$

$$G_{\lambda\lambda_1\ldots\lambda_p\nu_1\ldots\nu_q}(z_1, \ldots z_m, \bar{z}_1', \ldots \bar{z}_n'; Z, Z_1, \ldots Z_p, \bar{Z}_1', \ldots \bar{Z}_q')$$
$$= -\int d^4\mu(Z') \sum_{j=1}^{q} g_{\lambda\nu_j} D(Z, \bar{Z}')U^{(4)}(Z' - \bar{Z}_j')G_{\lambda_1\ldots\lambda_p\nu_1\ldots\nu_j\ldots\nu_q}z_1,$$
$$\ldots z_m, \bar{z}_1', \ldots \bar{z}_n'; Z_1, \ldots Z_p, \bar{Z}_1', \ldots \hat{\bar{Z}}_j', \ldots \bar{Z}_q')$$
$$-e\int \ldots \int d^4\mu(z')\, d^4\mu(z'')\, d^4\mu(Z')D(Z, \bar{Z}')W^{(4)}(Z', \bar{z}', z'')\gamma_\lambda$$
$$\times G_{\lambda_1\ldots\lambda_p\nu_1\ldots\nu_q}(z', z_1, \ldots z_m, \bar{z}'', \bar{z}_1', \ldots \bar{z}_n'; Z_1, \ldots Z_p, \bar{Z}_1', \ldots \bar{Z}_q'). \tag{12c}$$

If we now try to use these infinitely coupled equations any further, we run into evident complexity of the equations. Introduction of a suitable graphical representation would enable us to use these equations with more ease, and this is what we propose to do next.

Corresponding to a many-particle Green's function G, let us introduce a Γ-function defined by

$$\Gamma_{\lambda_1\ldots\lambda_p\nu_1\ldots\nu_q}(z_1, \ldots z_m, \bar{z}_1', \ldots \bar{z}_n'; Z_1, \ldots Z_p, \bar{Z}_1', \ldots \bar{Z}_q')$$
$$= (i\nabla_{z_1} + m) \ldots (i\nabla_{z_m} + m)\Box_{Z_1} \cdots \Box_{Z_p} G_{\lambda_1\ldots\lambda_p\nu_1\ldots\nu_q}(z_1,$$
$$\ldots z_m, \bar{z}_1', \ldots \bar{z}_n'; Z_1, \ldots Z_p, \bar{Z}_1', \ldots \bar{Z}_q')\Box_{\bar{Z}_q} \cdots \Box_{\bar{Z}_1'}(-i\nabla_{\bar{z}_n'} + m)$$
$$\ldots (-i\nabla_{\bar{z}_1'} + m). \tag{13a}$$

The inverse relationship between Γ and G can then be easily established

$$G_{\lambda_1\ldots\lambda_p\nu_1\ldots\nu_q}(z_1, \ldots z_m, \bar{z}_1', \ldots \bar{z}_n'; Z_1, \ldots Z_p, \bar{Z}_1', \ldots \bar{Z}_q')$$
$$= \int \ldots \int d^4\mu(y_1) \ldots d^4\mu(y_m)\, d^4\mu(y_1') \ldots d^4\mu(y_n')\, d^4\mu(Y_1) \ldots d^4\mu(Y_p)\, d^4\mu(Y_1')$$
$$\ldots d^4\mu(Y_q')\, S(z_1, \bar{y}_1) \ldots S(z_m, \bar{y}_m)D_{\lambda_1\lambda_1'}(Z_1, \bar{Y}_1) \ldots D_{\lambda_p\lambda_p'}(Z_p, \bar{Y}_p)$$
$$\times \Gamma_{\lambda_1'\ldots\lambda_p'\nu_1'\ldots\nu_q'}(y_1, \ldots y_m, \bar{y}_1', \ldots \bar{y}_n'; Y_1, \ldots Y_p, \bar{Y}_1', \ldots \bar{Y}_q')$$
$$\times D_{\nu_q'\nu_q}(Y_q', \bar{Z}_q') \ldots D_{\nu_1'\nu_1}(Y_1', \bar{Z}_1')\, S(y_n', \bar{z}_n') \ldots S(y_1', \bar{z}_1'). \tag{13b}$$

One can now use these relationships between G and Γ to obtain an infinite set of coupled equations from (12*a*):

$$\int d^4\mu(z')\,S(z,\bar{z}')\Gamma_{\lambda_1\ldots\lambda_p\nu_1\ldots\nu_q}(z',z_1,\ldots z_m,\bar{z}_1',\ldots\bar{z}_n';Z_1,\ldots Z_p,\bar{Z}_1',\ldots\bar{Z}_q')$$
$$=\int d^4\mu(z')\sum_{j=1}^{n}S(z,\bar{z}')U^{(4)}(z'-\bar{z}_j')(-i\nabla_{\bar{z}_j'}+m)\Gamma_{\lambda_1\ldots\lambda_p\nu_1\ldots\nu_q}(z_1,$$
$$\ldots z_m,\bar{z}_1',\ldots\hat{\bar{z}}_j',\ldots\bar{z}_n';Z_1,\ldots Z_p,\bar{Z}_1',\ldots\bar{Z}_q')+e\int\ldots\int d^4\mu(z')$$
$$\times d^4\mu(z'')\,d^4\mu(z''')\,d^4\mu(Z)\,d^4\mu(Z')\,S(z,\bar{z}')W^{(4)}(z',\bar{z}'',\bar{Z}')\gamma_\lambda$$
$$\times S(z'',\bar{z}''')D_{\lambda\lambda'}(Z',\bar{Z})\Gamma_{\lambda'\lambda_1\ldots\lambda_p\nu_1\ldots\nu_q}(z''',z_1,\ldots z_m,\bar{z}_1',\ldots\bar{z}_n';Z,Z_1,$$
$$\ldots Z_p,\bar{Z}_1',\ldots\bar{Z}_q'). \qquad (14a)$$

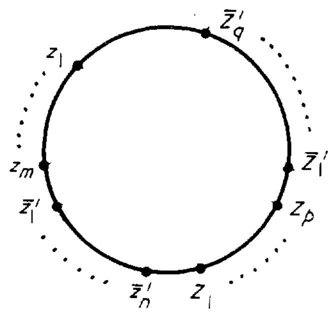
If we now use the following graphical representation

$S(z,\bar{z}')\equiv$

$D_{\lambda\nu}(Z,\bar{Z}')\equiv$

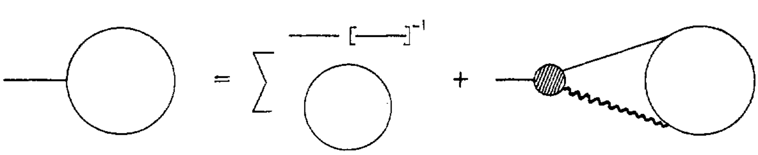
$\Gamma_{\lambda_1\cdots\lambda_p\nu_1\cdots\nu_q}(z_1,\cdots z_m,\bar{z}_1'\cdots\bar{z}_n';Z_1,\cdots Z_p,\bar{Z}_1',\cdots\bar{Z}_q')$

$\equiv$



$e\gamma_\lambda W^{(4)}(z,z',z'')\equiv$

then equation (14*a*) can be given a simple graphical representation



(14*b*)

where we have used

$$U^{(4)}(z-\bar{z}')(-i\nabla_{\bar{z}'}+m)=\{S(z,\bar{z}')\}^{-1} \qquad (14c)$$

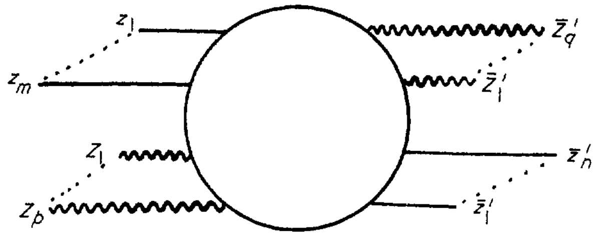
which satisfies the equation

$$\int d^4\mu(z'')\, S(z, \bar{z}'')\{S(z'', \bar{z}')\}^{-1} = U^{(4)}(z - \bar{z}') \tag{14d}$$

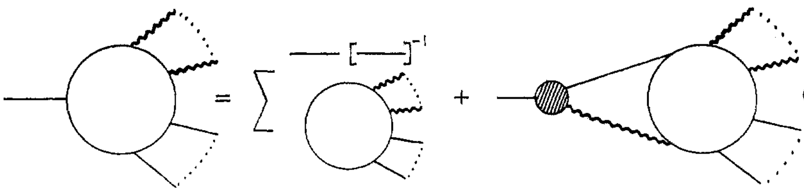
and it is being understood that we integrate over all the common points in the diagrams.

From (13*b*) one can see that according to the proposed graphical representation a many-particle Green's function will have the representation of a Γ-function with external electron and photon lines, that is

$$G_{\lambda_1 \ldots \lambda_p \nu_1 \ldots \nu_q}(z_1, \ldots z_m, \bar{z}_1', \ldots \bar{z}_n'; Z_1, \ldots Z_p, \bar{Z}_1', \ldots \bar{Z}_q')$$

≡ 

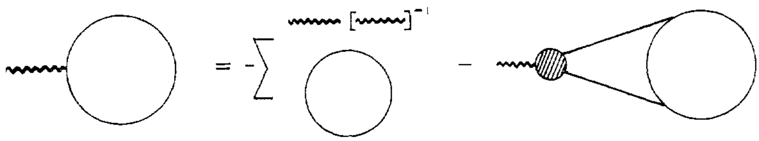
Applying (14*b*), which shows how to split up a Γ-function with one external electron line, to a many-particle Green's function we then get the following coupled equation:



(15*a*)

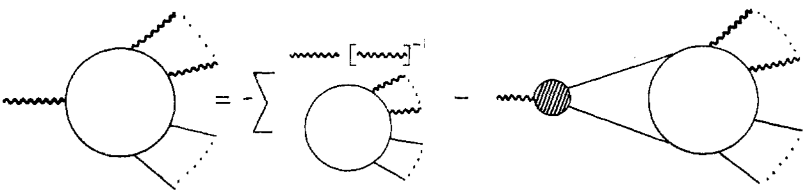
whose analytic form can be written down by the rules established for graphical representation.

Starting from (12*c*) and following the same technique as above, it can be shown that a Γ-function with an external photon line can be split up as follows:



(15*b*)

which gives another set of coupled equations for the many-particle Green's functions of electrodynamics



(15*c*)

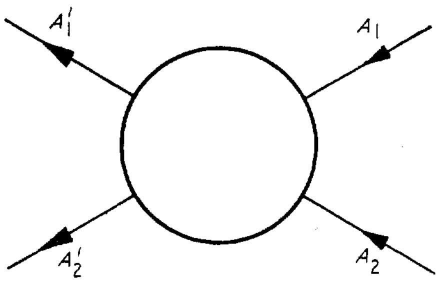
Equations (15*a*, *c*) are the infinitely coupled equations for the many-particle Green's functions in z-space and are same in structure as those in the conventional theory except for the presence of the interaction kernel W at each vertex.

4. Matrix elements for scattering processes

We next proceed to calculate the matrix elements for Compton and Møller scatterings from the z-space Green's functions. The relationship between the z-space Green's functions and the matrix elements for scattering processes, which is similar to the one in the conventional theory, can be established as follows.

Consider a two-particle Green's function, $G(z_1', z_2', \bar{z}_1, \bar{z}_2)$, in z-space. On multiplying the Γ-function corresponding to the Green's function G by the free particle wave functions $A_n(\boldsymbol{p}_n)\exp(-ip_n z_n)$ and $A_n'(\boldsymbol{p}_n')\exp(ip_n'\bar{z}_n')$, $n = 1, 2$, A_n, A_n' being the momentum space amplitudes of the particles involved in the scattering process, and integrating over z_n, z_n', we get

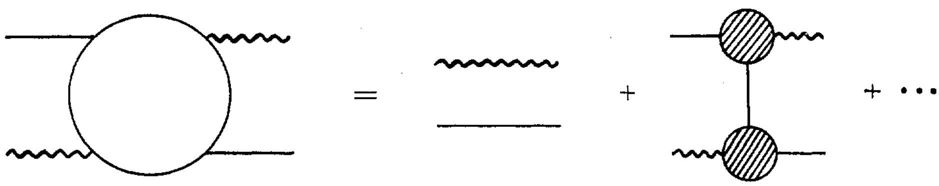
$$\int \ldots \int d^4\mu(z_1)\, d^4\mu(z_2)\, d^4\mu(z_1')\, d^4\mu(z_2') \exp\{-i(p_1 z_1 + p_2 z_2 - p_1'\bar{z}_1' - p_2'\bar{z}_2')\}$$
$$\times A_1'(\boldsymbol{p}_1')A_2'(\boldsymbol{p}_2')(i\nabla_{z_1'} + m)(i\nabla_{z_2'} + m)G(z_1', z_2', \bar{z}_1, \bar{z}_2)(-i\nabla_{\bar{z}_2} + m)$$
$$\times (-i\nabla_{\bar{z}_1} + m)A_2(\boldsymbol{p}_2)A_1(\boldsymbol{p}_1) \tag{16}$$



The quantity in (16) is the Fourier transform of the Γ-function corresponding to G multiplied by the momentum space amplitudes of the free particles which is the matrix element in momentum space for the scattering of free incoming particles A_1 and A_2 into free outgoing particles A_1' and A_2' (Gasiorowicz 1966) as shown in the diagram. The diagram can be expanded using the coupled equations of the previous section to give all the diagrams of all orders that contribute to the scattering process and the matrix elements for different orders can be calculated according to the prescription given above.

4.1. *Compton scattering*

In the expansion of a two-particle Green's function $G(z, \bar{z}'; Z, \bar{Z}')$



$$G \quad = \quad G^{(0)} \quad + \quad G^{(2)} \quad + \cdots$$

the second-order diagram can be written down analytically as

$$G^{(2)}(z, \bar{z}'; Z, \bar{Z}') = e^2 \int \ldots \int d^4\mu(z_1) \ldots d^4\mu(z_6)\, S(z, \bar{z}_1) W^{(4)}(z_1, z_2, \bar{z}_3) D(Z, \bar{z}_2)$$
$$\times \gamma_\lambda S(z_3, \bar{z}_4)\gamma_\nu D(z_5, \bar{Z}') W^{(4)}(z_4, \bar{z}_5, \bar{z}_6)\, S(z_6, \bar{z}').$$

Then with the prescription given above, the matrix element for second-order Compton scattering becomes

$$M_{e\gamma}^{(2)}(p_1, p_2, k_1, k_2) = e^2 \int \dots \int \mathrm{d}^4\mu(z_1) \dots \mathrm{d}^4\mu(z_6) \exp\{-\mathrm{i}(p_1 z_6 + k_1 z_5 - p_2 \bar{z}_1 - k_2 \bar{z}_2)\}$$
$$\times \bar{\Psi}^r(\boldsymbol{p}_2) A_\lambda(\boldsymbol{k}_2) W^{(4)}(z_1, z_2, \bar{z}_3)\gamma_\lambda \mathrm{S}(z_3, \bar{z}_4)\gamma_\nu$$
$$\times W^{(4)}(z_4, \bar{z}_5, \bar{z}_6)\, A_\nu(\boldsymbol{k}_1)\, \Psi^r(\boldsymbol{p}_1). \tag{17}$$

From the connection between the momentum spaces of q-space and z-space established in I, the momentum space amplitudes of the free electron and photon wave functions are found to be

$$\Psi^r(\boldsymbol{p}) = \frac{4\pi a^2}{(2\pi)^{3/2}} \left(\frac{m}{E(\boldsymbol{p})}\right)^{1/2} \exp(-\tfrac{1}{2}a^2 p^2) w^r(\boldsymbol{p}) \tag{18a}$$

$$\bar{\Psi}^r(\boldsymbol{p}) = \frac{4\pi a^2}{(2\pi)^{3/2}} \left(\frac{m}{E(\boldsymbol{p})}\right)^{1/2} \exp(-\tfrac{1}{2}a^2 p^2) \bar{w}^r(\boldsymbol{p}) \tag{18b}$$

$$A_\lambda(k) = \frac{4\pi a^2}{2(2\pi)^{3/2}} \frac{1}{\omega(\boldsymbol{k})} \exp(-\tfrac{1}{2}a^2 k^2)\, \epsilon_\lambda(\boldsymbol{k}). \tag{18c}$$

Using the Fourier transform of the interaction kernel W

$$W^{(4)}(z, z', \bar{z}'') = \frac{1}{(2\pi)^{12}} \int \dots \int \mathrm{d}^4 p\, \mathrm{d}^4 p'\, \mathrm{d}^4 p'' \exp\{-\mathrm{i}(pz + p'z' - p''\bar{z}'')\} W^{(4)}(p, p', p'') \tag{19a}$$

$$W^{(4)}(p, p', p'') = (2\pi)^4 (4\pi a^2)^3 \delta^{(4)}(p + p' - p'') \hat{W}^{(4)}(p, p', p'') \tag{19b}$$

$$\hat{W}^{(4)}(p, p', p'') = \exp\{-a^2(p''^2 - pp')\} \tag{19c}$$

and the Fourier transform of the free electron Green's function S

$$\mathrm{S}(z, \bar{z}') = \frac{1}{(2\pi)^4} \int \mathrm{d}^4 p \exp\{-\mathrm{i}p(z - \bar{z}')\} \mathrm{S}(p) \tag{19d}$$

$$\mathrm{S}(p) = (4\pi a^2)^2 \exp(-a^2 p^2) \frac{\not{p} + m}{p^2 + m^2} \tag{19e}$$

the z-integrations in (17) can be done with the help of the identity

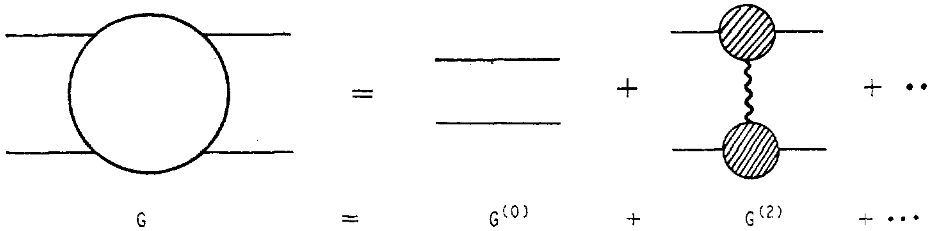
$$\int \mathrm{d}^4\mu(z) \exp\{-\mathrm{i}(\alpha z - \beta \bar{z})\} = \frac{(2\pi)^4}{(4\pi a^2)^3} \delta^{(4)}(\alpha - \beta) \exp\left\{a^2 \left(\frac{\alpha + \beta}{2}\right)^2\right\}$$
$$= \frac{(2\pi)^4}{(4\pi a^2)^3} \delta^{(4)}(\alpha - \beta) \exp(a^2 \alpha^2) \tag{20}$$

and the remaining integrations can be done using the δ-functions arising out of the z-integrations to give

$$M_{e\gamma}^{(2)}(p_1, p_2, k_1, k_2) = \frac{e^2}{8\pi^2} \left(\frac{m^2}{E(\boldsymbol{p}_1)E(\boldsymbol{p}_2)\omega(\boldsymbol{k}_1)\omega(\boldsymbol{k}_2)}\right)^{1/2} \delta^{(4)}(p_2 + k_2 - p_1 - k_1)$$
$$\times \exp\{a^2(p_1^2 + k_1^2 + p_1 k_1 + p_2^2 + k_2^2 + p_2 k_2)\}$$
$$\times \hat{W}^{(4)}(p_2, k_2, p_2 + k_2) \bar{w}^{r_2}(\boldsymbol{p}_2) \not{\epsilon}(\boldsymbol{k}_2) \frac{\not{p}_2 + \not{k}_2 + m}{(p_2 + k_2)^2 + m^2} \not{\epsilon}(\boldsymbol{k}_1) w^{r_1}(\boldsymbol{p}_1)$$
$$\times \hat{W}^{(4)}(p_1, k_1, p_1 + k_1). \tag{21}$$

4.2. *Møller scattering*

The matrix element for second-order Møller scattering can be calculated in the same way from a two-electron Green's function $G(z, z', \bar{z}'', \bar{z}''')$ in z-space. From the second-order term in the expansion of G



one gets the matrix element for second-order Møller scattering

$$M_{ee}^{(2)}(p_1, p_2, P_1, P_2) = \int \ldots \int d^4\mu(z)\, d^4\mu(z')\, d^4\mu(z'')\, d^4\mu(z''')\exp\{i(P_1\bar{z}+P_2\bar{z}')\}$$
$$\times\bar{\Psi}(\boldsymbol{P}_1)\bar{\Psi}(\boldsymbol{P}_2)(i\nabla_z+m)(i\nabla_{z'}+m)G^{(2)}(z, z', \bar{z}'', \bar{z}''')$$
$$\times(-i\nabla_{\bar{z}'''}+m)(-i\nabla_{\bar{z}''}+m)\Psi(\boldsymbol{p}_2)\Psi(\boldsymbol{p}_1)\exp\{-i(p_1 z''+p_2 z''')\} \quad (22a)$$

where

$$G^{(2)}(z, z', \bar{z}'', \bar{z}''') = e^2\int \ldots \int d^4\mu(z_1) \ldots d^4\mu(z_6)\, S(z, \bar{z}_1)W^{(4)}(z_1, \bar{z}_2, \bar{z}_3)\gamma_\lambda$$
$$\times S(z_2, \bar{z}'')D_{\lambda\nu}(z_3, \bar{z}_4)\, S(z', \bar{z}_5)\gamma_\nu W^{(4)}(z_4, z_5, \bar{z}_6)\, S(z_6, \bar{z}'''). \quad (22b)$$

Once, again, using the Fourier transform of the interaction kernel W, the electron Green's function S, the Fourier transform of the free photon Green's function D,

$$D(z, \bar{z}') = \frac{1}{(2\pi)^4}\int d^4k \exp\{-ik(z-\bar{z}')\}D(k) \quad (22c)$$

$$D(k) = (4\pi a^2)^2\exp(-a^2k^2)\frac{1}{k^2} \quad (22d)$$

and the identity (20), one gets from (22a)

$$M_{ee}^{(2)}(p_1, p_2, P_1, P_2) = \frac{e^2}{4\pi^2}\left(\frac{m^4}{E(\boldsymbol{p}_1)E(\boldsymbol{p}_2)E(\boldsymbol{P}_1)E(\boldsymbol{P}_2)}\right)^{1/2}\delta^{(4)}(P_1+P_2-p_1-p_2)$$
$$\times\exp\{a^2(P_1^2+p_1^2-P_1p_1+P_2^2+p_2^2-P_2p_2)\}$$
$$\times\hat{W}^{(4)}(P_1-p_1, p_1, P_1)$$
$$\times\frac{\{\bar{w}^{r_1}(\boldsymbol{P}_1)\gamma_\lambda w^{s_1}(\boldsymbol{p}_1)\}\{\bar{w}^{r_2}(\boldsymbol{P}_2)\gamma_\lambda w^{s_2}(\boldsymbol{p}_2)\}}{(P_2-p_2)^2}$$
$$\times\hat{W}^{(4)}(P_2-p_2, p_2, P_2). \quad (23)$$

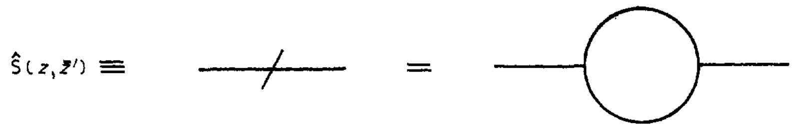
Equations (21) and (23) give the matrix elements for second-order Compton and Møller scatterings respectively with an interaction kernel W. With the choice of (19a, b, c) for W, which corresponds to a local interaction in q-space, and a transformation back to Lorentz metric, these are exactly the same as the matrix elements obtained from the conventional theory (Schweber 1961).

5. Divergent diagrams, regularization and nonlocal interactions

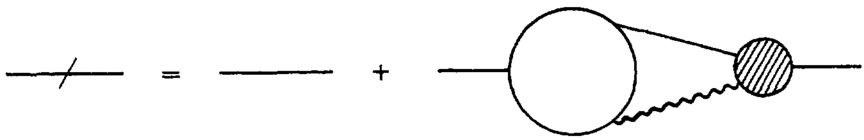
We next look into the basic divergent diagrams of electrodynamics, namely, electron self-energy, vacuum polarization and the vertex operator. To calculate the electron self-energy from z-space formulation, we start with the total electron Green's function $\hat{S}$ which satisfies the equation

$$\hat{S}(z, \bar{z}') = S(z, \bar{z}') + \int\int d^4\mu(z_1)\, d^4\mu(z_2)\, S(z, \bar{z}_1)\Sigma(z_1, \bar{z}_2)\, \hat{S}(z_2, \bar{z}'). \qquad (24a)$$

With the graphical notation



we get the exact equation for the total electron Green's function,

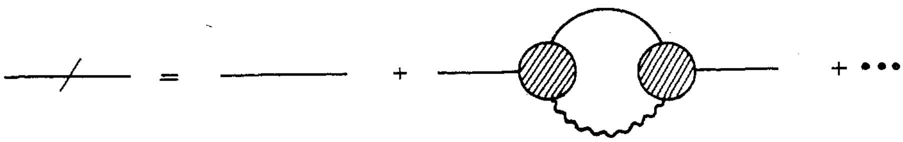


(24b)

with the analytical form

$$\hat{S}(z, \bar{z}') = S(z, \bar{z}') + e\int \dots \int d^4\mu(z_1) \dots d^4\mu(z_6)\, S(z, \bar{z}_1)\Gamma_\lambda(z_1, \bar{z}_2, \bar{z}_3)$$
$$\times D_{\lambda\nu}(z_2, \bar{z}_4)\, S(z_3, \bar{z}_5)\gamma_\nu W^{(4)}(z_4, z_5, \bar{z}_6)\, S(z_6, \bar{z}'). \qquad (24c)$$

In the expansion of the total electron Green's function $\hat{S}$



the second-order term has the analytic form

$$S^{(2)}(z, \bar{z}') = \int\int d^4\mu(z_1)\, d^4\mu(z_2)\, S(z, \bar{z}_1) \sum (z_1, \bar{z}_2)\, S(z_2, \bar{z}') \qquad (25)$$

where Σ is the second-order electron self-energy

$$\sum (z, \bar{z}') = e^2 \int \dots \int d^4\mu(z_1) \dots d^4\mu(z_4)\, W^{(4)}(z, \bar{z}_1, \bar{z}_2)\gamma_\lambda\, S(z_1, \bar{z}_3)\gamma_\nu$$
$$\times D_{\lambda\nu}(z_2, \bar{z}_4)\, W^{(4)}(z_3, z_4, \bar{z}'). \qquad (26)$$

With the proper-time parameter representation of S and D (different representations of S and D have been worked out in I)

$$S(z, \bar{z}') = i\int_0^\infty ds\, S(z, \bar{z}'; s) \qquad (27a)$$

$$S(z, \bar{z}'; s) = \frac{1}{(2\pi)^4}\int d^4p\, \exp\{-ip(z-\bar{z}')\}(\not{p}+m)\Delta(p; \mu^2 = m^2; s) \qquad (27b)$$

and

$$D(z, \bar{z}') = i\int_0^\infty ds\, D(z, \bar{z}'; s) \tag{27c}$$

$$D(z, \bar{z}'; s) = \frac{1}{(2\pi)^4}\int d^4k \exp\{-ik(z-\bar{z}')\}\Delta(k; \mu^2 = 0; s) \tag{27d}$$

where

$$\Delta(k; \mu^2; s) = (4\pi a^2)^2 \exp\{-i(k^2+\mu^2)s\} \exp(-a^2k^2) \tag{27e}$$

and the Fourier transform of W, one can reduce (26) to get

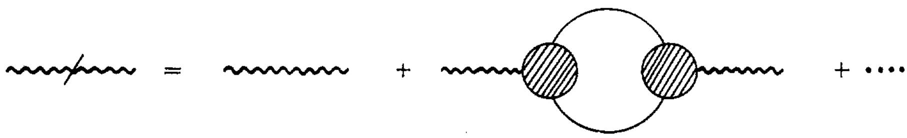
$$\sum(z, \bar{z}') = \frac{1}{(2\pi)^4}\int d^4p \exp\{-ip(z-\bar{z}')\} \sum(p) \tag{28a}$$

$$\begin{aligned}\sum(p) = &-\frac{e^2}{(2\pi)^4}(4\pi a^2)^2 \int_0^\infty ds_1 \int_0^\infty ds_2 \int d^4k \exp[a^2\{(p-k)^2+k^2\}] \\ &\times \exp[-i\{(p-k)^2+m^2\}s_1] \exp(-ik^2s_2)\hat{W}^{(4)}(p-k, k, p) \\ &\times \gamma_\lambda(\not{p}-\not{k}+m)\gamma_\lambda \hat{W}^{(4)}(k, p-k, p).\end{aligned} \tag{28b}$$

Next we consider vacuum polarization. The total photon Green's function D,

$\hat{D}_{\lambda\nu}(z, \bar{z}') = G_{\lambda\nu}(z, \bar{z}') \equiv$ 

has the expansion



in which the second-order diagram contains the vacuum polarization term

$$\begin{aligned}\Pi^{\lambda\nu}(z, \bar{z}') = e^2 \int \dots \int d^4\mu(z_1) \dots d^4\mu(z_4)\, W^{(4)}(z, z_1, \bar{z}_2) \\ \times \mathrm{Tr}\{\gamma_\lambda S(z_3, \bar{z}_1)\gamma_\nu S(z_2, \bar{z}_4)\} W^{(4)}(\bar{z}_3, z_4, \bar{z}')\end{aligned} \tag{29}$$

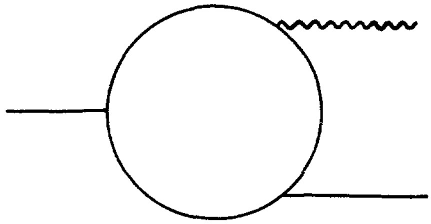
which can be reduced in the same way as before to give

$$\Pi^{\lambda\nu}(z, \bar{z}') = \frac{1}{(2\pi)^4}\int d^4k \exp\{-ik(z-\bar{z}')\}\, \Pi^{\lambda\nu}(k) \tag{30a}$$

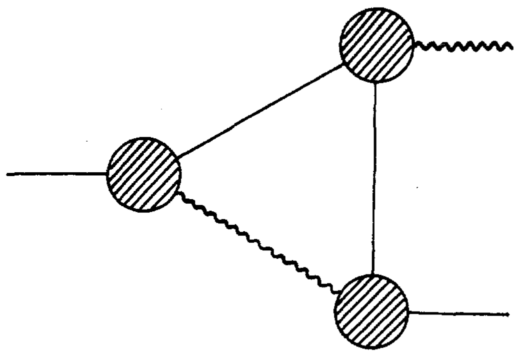
with

$$\begin{aligned}\Pi^{\lambda\nu}(k) = &-\frac{e^2}{(2\pi)^4}(4\pi a^2)^2 \int_0^\infty ds_1 \int_0^\infty ds_2 \int d^4p \exp[a^2\{(p+k)^2+p^2\}] \\ &\times \exp[-i(p^2+m^2)s_1] \exp[-i\{(p+k)^2+m^2\}s_2]\hat{W}^{(4)}(p, k, p+k) \\ &\times \mathrm{Tr}[\gamma_\lambda(\not{p}+m)\gamma_\nu(\not{p}+\not{k}+m)]\hat{W}^{(4)}(p, k, p+k).\end{aligned} \tag{30b}$$

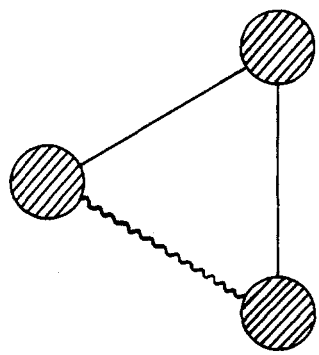
Finally we consider the vertex operator. In the coupled equations for the many-particle Green's functions there is always a vertex part



which in its lowest order reduces to



The analytical form of the vertex operator



is

$$\Lambda^{\nu}(z, \bar{z}', \bar{z}'') = e^2 \int \dots \int \mathrm{d}^4\mu(z_1) \dots \mathrm{d}^4\mu(z_6) W^{(4)}(z, \bar{z}_1, \bar{z}_2)\gamma_\lambda \mathrm{S}(z_1, \bar{z}_3)$$
$$\times \mathrm{D}_{\lambda\delta}(z_2, \bar{z}_5)\gamma_\nu \mathrm{S}(z_4, \bar{z}_6) W^{(4)}(z_3, \bar{z}_4, \bar{z}')\gamma_\delta W^{(4)}(z_5, z_6, \bar{z}'') \qquad (31a)$$

and when reduced, gives the momentum space representation of the vertex operator Λ to the order α,

$$\Lambda^{\nu}(p_1, p_2) = -\frac{\mathrm{i}\alpha}{4\pi^3}(4\pi a^2)^3 \int_0^\infty \mathrm{d}s_1 \int_0^\infty \mathrm{d}s_2 \int_0^\infty \mathrm{d}s_3 \int \mathrm{d}^4k \exp[a^2\{(p_1-k)^2$$
$$+(p_2-k)^2+k^2\}]$$
$$\times \exp[-\mathrm{i}\{(p_1-k)^2+m^2\}s_1] \exp[-\mathrm{i}\{(p_2-k)^2+m^2\}s_2] \exp(-\mathrm{i}k^2 s_3)$$
$$\times \hat{W}^{(4)}(p_1-k, k, p_1)\gamma_\lambda(\not{p}_1-\not{k}+m)\gamma_\nu \hat{W}^{(4)}(p_1-p_2, p_2-k, p_1-k)$$
$$\times (\not{p}_2-\not{k}+m)\gamma_\lambda \hat{W}^{(4)}(k, p_2-k, p_2). \qquad (31b)$$

If one now chooses W given by (19a, b, c) corresponding to a local interaction in q-space, equations (28b), (30b) and (31b) reduce to the standard divergent expressions for $\Sigma(p)$, $\Pi^{\lambda\nu}(k)$ and $\Lambda^{\nu}(p_1, p_2)$ of the conventional theory, apart from some normalization factors, and one can regularize these divergent expressions using any standard regularization technique.

The interesting feature of the analytical representation of quantum field theory is that a change in the structure of the interaction kernel W does not change the formulation of the theory since the form of the field equations, of the coupled equations for the many-particle Green's functions and of the Green's functions themselves remains unchanged. But any change in W will correspond to a nonlocal interaction in q-space and will, therefore, affect the observables. Thus one can consider the analytic representation of quantum field theory as a more general theory with the local q-space theory built in with the particular choice of W given by (6e). It will be of interest then to look into the different structures of W and study their effects on the observables. We now propose to show that a suitable modification of the interaction kernel W makes the basic divergences of electrodynamics convergent and can be used as a regularization technique.

It is obvious that in choosing the modified interaction kernel we should be guided by the following requirements:

(i) the modified interaction kernel must be a symmetric function of three sets of complex variables, being analytic in each of the three sets of variables, so that the general formulation of the theory will remain unchanged;

(ii) the Fourier transform of the modified interaction kernel must ensure energy-momentum conservation at each nonlocal vertex in order that the theory will be physically realistic;

(iii) under a suitably defined limiting process the modified interaction kernel must reduce to the original interaction kernel (6e) corresponding to the local q-space interaction.

In principle, one can construct a large number of modified interaction kernels which will meet the above requirements. For our discussion we choose the following modified interaction kernel:

$$W_\beta^{(4)}(z, z', z'') = [\pi\{\tfrac{3}{2}(1+\beta)^2 a\}^2]^{-1} \exp\left(-\frac{(z-z')^2+(z'-z'')^2+(z''-z)^2}{6(1+\beta)a^2}\right) \tag{32}$$

where β is a dimensionless parameter.

The above kernel is analytic in each of the three sets of complex variables satisfying the first requirement. The Fourier transform of this modified kernel turns out to be

$$W_\beta^{(4)}(z, z', z'') = \frac{1}{(2\pi)^{12}} \int \cdots \int \mathrm{d}^4p\, \mathrm{d}^4p'\, \mathrm{d}^4p'' \exp\{-\mathrm{i}(pz+p'z'-p''\bar{z}'')\} \times W_\beta^{(4)}(p, p', p'') \tag{33a}$$

$$W_\beta^{(4)}(p, p', p'') = (2\pi)^4(4\pi a^2)^3 \delta^{(4)}(p+p'-p'')\hat{W}_\beta^{(4)}(p, p', p'') \tag{33b}$$

$$\hat{W}_\beta^{(4)}(p, p', p'') = \exp\{-(1+\beta)a^2(p''-pp')\} \tag{33c}$$

which assures energy–momentum conservation at each nonlocal vertex. In the limit $\beta \to 0$, one has

$$\lim_{\beta \to 0} W_\beta \to W$$

W being the interaction kernel (6e) corresponding to the local interaction in q-space. The modified interaction kernel thus meets all the requirements.

If one now calculates the electron self-energy Σ, defined by (26), using the modified interaction kernel, one gets in momentum space representation

$$\sum_\beta(p) = -\frac{e^2}{(2\pi)^4}(4\pi a^2)^2 \exp(-a^2p^2)\int_0^\infty ds_1 \int_0^\infty ds_2 \int d^4k\gamma_\lambda(\not{p}-\not{k}+m)\gamma_\lambda$$
$$\times \exp[-i\{(p-k)^2+m^2\}s_1]\exp(-ik^2s_2)\exp\{-2\beta a^2(p^2+k^2-pk)\}$$

which, after a rotation of the s_1 and s_2 paths of integration through $-\frac{1}{2}\pi$ and a translation of the origin, takes the form

$$\sum_\beta(p) = \frac{e^2}{(2\pi)^4}(4\pi a^2)^2 \exp\{-(1+\beta)a^2p^2\}\exp(\beta a^2m^2)$$
$$\times \int_{\beta a^2}^\infty ds_1 \int_{\beta a^2}^\infty ds_2 \int d^4k\gamma_\lambda(\not{p}-\not{k}+m)\gamma_\lambda \exp[-\{(p-k)^2+m^2\}s_1]$$
$$\times \exp(-ik^2s_2). \tag{34}$$

Similarly, one has for vacuum polarization

$$\Pi_\beta{}^{\lambda\nu}(k) = \frac{e^2}{(2\pi)^4}(4\pi a^2)^2 \exp\{-(1+\beta)a^2k^2\}\exp(2\beta a^2m^2)$$
$$\times \int_{\beta a^2}^\infty ds_1 \int_{\beta a^2}^\infty ds_2 \int d^4p \,\mathrm{Tr}\{\gamma_\lambda(\not{p}+m)\gamma_\nu(\not{p}+\not{k}+m)\}$$
$$\times \exp\{-(p^2+m^2)s_1\}\exp[-\{(p+k)^2+m^2\}s_2] \tag{35}$$

and for the vertex operator to the order α

$$\Lambda_\beta{}^\nu(p_1, p_2) = \frac{\alpha}{4\pi^3}(4\pi a^2)^3 \exp\{-(1+\beta)a^2(p_1{}^2+p_2{}^2-p_1p_2)\}\exp(2\beta a_2m^2)$$
$$\times \int_{\beta a^2}^\infty ds_1 \int_{\beta a^2}^\infty ds_2 \int_{\beta a^2}^\infty ds_3 \int d^4k\gamma_\lambda(\not{p}_1-\not{k}+m)\gamma_\nu(\not{p}_2-\not{k}+m)\gamma_\lambda$$
$$\times \exp[-\{(p_1-k)^2+m^2\}s_1]\exp[-\{(p_2-k)^2+m^2\}s_2]\exp(-k^2s_3). \tag{36}$$

A comparison of Σ_β, $\Pi_\beta{}^{\lambda\nu}$ and $\Lambda_\beta{}^\nu$ with Σ, $\Pi^{\lambda\nu}$ and Λ^ν confirms that

$$\lim_{\beta\to 0}\Sigma_\beta,\ \Pi_\beta{}^{\lambda\nu},\ \Lambda_\beta{}^\nu \to \Sigma,\ \Pi^{\lambda\nu},\ \Lambda^\nu.$$

Thus, the modification of the interaction kernel results in replacing the zero lower limits of the proper-time integrals in (28b), (30b) and (31b) by a nonzero lower limit βa^2. Since the divergences in Σ, $\Pi^{\lambda\nu}$ and Λ^ν occur in the final stages of integrating the proper time parameters to the origin, as long as $\beta > 0$, that is, as long as $\beta a^2 > 0$, the quantities Σ_β, $\Pi_\beta{}^{\lambda\nu}$ and $\Lambda_\beta{}^\nu$ are all finite. For negative values of β the integrals will diverge again, since for $\beta a^2 < 0$ the integration path has to pass through the origin where the divergences occur. Therefore, as long as the dimensionless parameter β is chosen to be positive, Σ_β, $\Pi_\beta{}^{\lambda\nu}$ and $\Lambda_\beta{}^\nu$ will be finite.

Schwinger (1951) has discussed a regularization process in which the zero lower limit of the proper-time integrals is replaced by a nonzero positive quantity s_0.

This makes the integrals convergent and the limit $s_0 \to 0$ is taken after suitable subtractions. In the calculations with the modified interaction, taking the limit $\beta \to 0$, which is equivalent to reducing the modified interaction kernel W_β to the original interaction kernel W corresponding to the local q-space interaction, makes the lower limit βa^2 of the proper-time integrals in (34), (35) and (36) go to zero. Thus the modification of the interaction kernel results in a regularization process similar to the one of Schwinger. We would like to point out though that the cut-off βa^2 is not an arbitrary one, but comes in naturally from the modified interaction.

We now examine the consequences of modifying the interaction kernel in q-space. The action due to the interaction term with the modified interaction kernel

$$I_\beta = -e \int \dots \int \mathrm{d}^4\mu(z)\,\mathrm{d}^4\mu(z')\,\mathrm{d}^4\mu(z'')W_\beta^{(4)}(z, \bar{z}', \bar{z}'')\bar{\Psi}(\bar{z})\gamma_\lambda \Psi(z')A_\lambda(z'') \tag{37}$$

corresponds to an interaction in q-space defined by the action integral

$$I_\beta = -e \int \dots \int \mathrm{d}^4q\,\mathrm{d}^4q'\,\mathrm{d}^4q''\,F_\beta(q, q', q'')\bar{\Psi}(q)\gamma_\lambda \Psi(q')\,A_\lambda(q'') \tag{38a}$$

where

$$F_\beta(q, q', q'') = \left(\frac{\pi^2}{3}\right)^{-2} (6\beta a^2)^{-4} \exp\left(-\frac{(q-q')^2+(q'-q'')^2+(q''-q)^2}{6\beta a^2}\right). \tag{38b}$$

The modification of the interaction kernel thus leads to a nonlocal interaction in q-space defined by (38a, b) with the q-space field equations modified to

$$(\mathrm{i}\nabla_q + m)\,\Psi(q) = e \int\int \mathrm{d}^4q'\,\mathrm{d}^4q''\,F_\beta(q, q', q'')\gamma_\lambda \Psi(q')\,A_\lambda(q'') \tag{39}$$

and to two similar equations.

Using the following representation of the δ-function,

$$\lim_{\epsilon \to 0} f_\epsilon^\alpha(q-q') = \delta(q-q')$$

$$f_\epsilon^\alpha(q) = \left(\frac{\alpha}{\pi\epsilon}\right)^{1/2} \exp\left(-\frac{\alpha q^2}{\epsilon}\right)$$

one finds that

$$\lim_{\beta \to 0} F_\beta(q, q', q'') = \delta^{(4)}(q-q')\delta^{(4)}(q-q''). \tag{40}$$

The modification of the interaction kernel therefore shows the possibility of formulating a nonlocal theory in q-space in which Σ_β, $\Pi_\beta^{\lambda\nu}$ and Λ_β^ν are finite. The action integral (38a) satisfies the general requirements of a nonlocal field theory proposed by Chretien and Peierls (1954). Since β is a dimensionless parameter and a^2 has the dimension of square of length, βa^2 may be considered to be the measure of the fundamental length associated with a nonlocal theory. To summarize, this modification of the interaction kernel leads to a regularizing process and also leads to the explicit form of a nonlocal interaction in q-space associated with the cut-off parameter of the regularization process.

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